

Celestial mechanics: Three-body problem

1 Introduction

Simulating (or determining) the movement of three (celestial) bodies with given parameters (masses, gravitational constant, initial conditions for the bodies' coordinates and velocities) is known as the *three-body problem*.¹ The first one to tackle this problem was Sir ISAAC NEWTON in the first book, section XI, of his *Principia*. According to Sir GEORGE BIDDELL AIRY, this is “the most valuable chapter that was ever written on physical science”.²

Only in the more simpler case of two bodies, an analytical solution is possible. Nevertheless, JOSEPH-LOUIS LAGRANGE was able to give some particular solutions for the much more difficult three-body problem. Since these are beyond the scope of this application note, the reader is referred to [BATTIN, 1999, pp. 365 ff.] and [MOULTON, 1923, pp. 277 ff.] for an analytical treatment of this problem. Using an analog computer, it is possible to examine the three-body problem.

2 Mathematical description

In the following, a simple universe, consisting of two suns and a satellite crisscrossing between these two heavenly bodies will be simulated using an Analog Paradigm analog computer. Such a system can be readily

¹This is a special case of the n -body problem which will not be investigated further in the following.

²Cf. [MOULTON, 1923, p. 363].

described in general by the following set of coupled differential equations:

$$\begin{aligned} m_1 \ddot{\vec{r}}_1 &= Gm_1m_2 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2^2 - \vec{r}_1^2|^3} + Gm_1m_3 \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3^2 - \vec{r}_1^2|^3} \\ m_2 \ddot{\vec{r}}_2 &= Gm_2m_3 \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3^2 - \vec{r}_2^2|^3} + Gm_2m_1 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1^2 - \vec{r}_2^2|^3} \\ m_3 \ddot{\vec{r}}_3 &= Gm_3m_1 \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1^2 - \vec{r}_3^2|^3} + Gm_3m_2 \frac{\vec{r}_2 - \vec{r}_3}{|\vec{r}_2^2 - \vec{r}_3^2|^3} \end{aligned} \quad (1)$$

Here, the vectors $\vec{r}_i$, $1 \leq i \leq 3$, describe the positions of the i bodies with masses m_i , while G is the gravitational constant.

To simplify things a bit, our universe is restricted to two dimensions. Further, the suns are assumed to have masses $m_1 = m_2 = 1$ while the third body, the *satellite*, is assumed to be of infinitesimal mass. The suns' positions $\vec{r}_1 = (x_1, y_1)$ and $\vec{r}_2 = (x_2, y_2)$ are fixed with $x_2 = -x_1$ and $y_1 = y_2 = 0$, thus only the satellite moves.

Accordingly, only equation (1) has to be implemented on the analog computer. Since the suns' masses are equal to 1, the satellite's mass cancels out, yielding

$$\ddot{\vec{r}}_3 = \frac{G}{|\vec{r}_1^2 - \vec{r}_3^2|^3}(\vec{r}_1 - \vec{r}_3) + \frac{G}{|\vec{r}_2^2 - \vec{r}_3^2|^3}(\vec{r}_2 - \vec{r}_3). \quad (2)$$

Introducing two distance terms

$$\Delta r_{13}^3 = \sqrt{(x_1 - x_3)^2 + y_3^2}^3 \quad \text{and} \quad (3)$$

$$\Delta r_{23}^3 = \sqrt{(x_2 - x_3)^2 + y_3^2}^3 \quad (4)$$

and splitting (2) into its x - and y -components yields the following two equations describing the satellite's trajectory in Cartesian coordinates:

$$\ddot{x}_3 = (x_1 - x_3) \frac{G}{\Delta r_{13}^3} + (x_2 - x_3) \frac{G}{\Delta r_{23}^3} \text{ and} \quad (5)$$

$$\ddot{y}_3 = -y_3 \left(\frac{G}{\Delta r_{13}^3} + \frac{G}{\Delta r_{23}^3} \right). \quad (6)$$

3 Computer setup

These equations can now be implemented on an analog computer: Figure 1 shows the partial computer setup yielding the distance terms (3) and (4). Instead of the two square root function generators followed by two multipliers, the required powers of $\frac{3}{2}$ could also be implemented by two diode function generators suitably set up, thus saving some computing elements.

Mechanizing (5) and (6) is pretty straightforward, too, as shown in figure 2. Since $y_1 = y_2 = 0$, the generation of y_3 does not require complex distance terms like the partial circuit yielding x_3 simplifying the overall setup considerably.

4 Results

Running this problem on an analog computer is fascinating due to the chaotic behaviour of the satellite's trajectory. Figure 3 shows a family of trajectories of the satellite with the analog computer running in repetitive mode of operation. The time scale factor k_0 of the integrators was set to

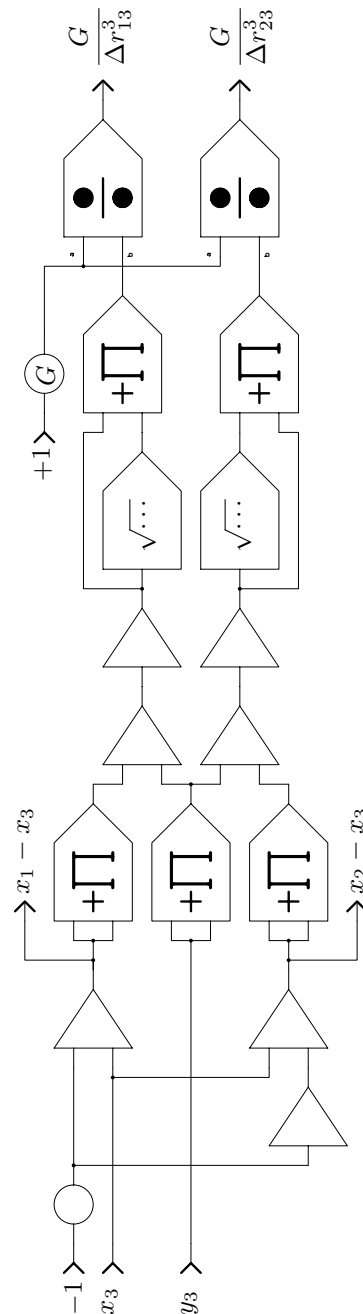


Figure 1: Distance terms

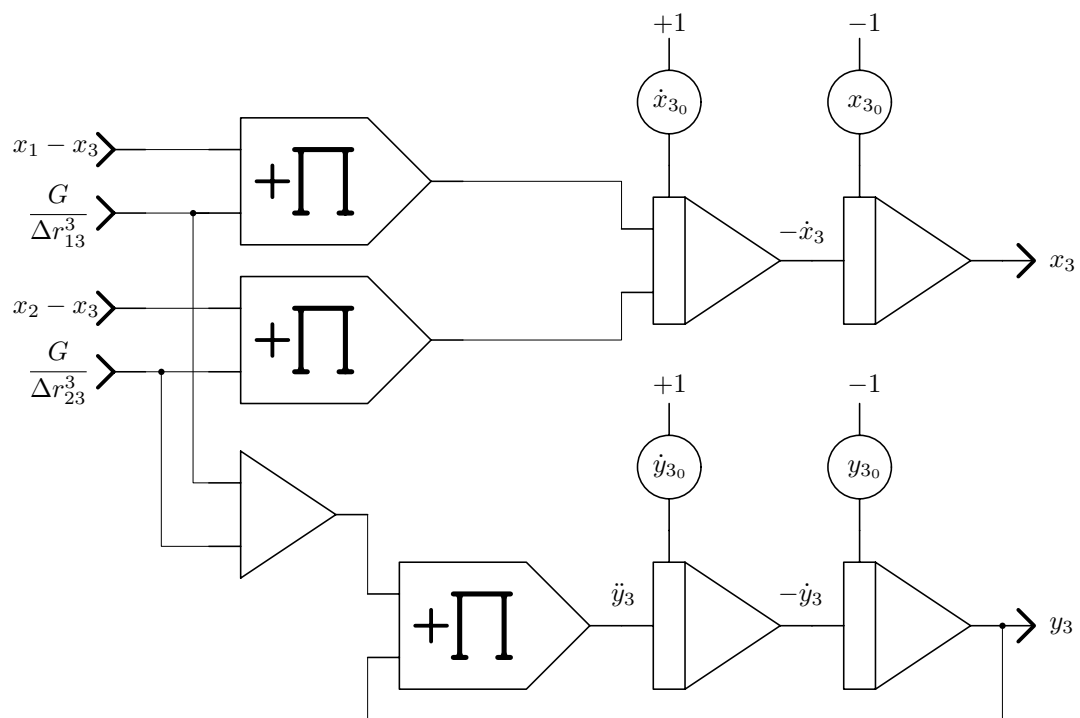


Figure 2: Implementation of equations (5) and (6)



Analog Computer Applications

10^5 , the operation time was 10 ms. The initial conditions were $x_1 = 0.5$, $x_2 = -0.5$, $x_{3_0} = 0.1$, and $y_{3_0} = 0.2$. Figure 4 shows a similar family of trajectories for a larger value of G .

References

- [BATTIN, 1999] RICHARD H. BATTIN, *An introduction to the mathematics and methods of astrodynamics*, American Institute of Aeronautics and Astronautics, Inc., Reston, Virginia, 1999
- [MOULTON, 1923] FOREST RAY MOULTON, *An Introduction to Celestial Mechanics*, The Macmillan Company, Second Revised Edition, 1923

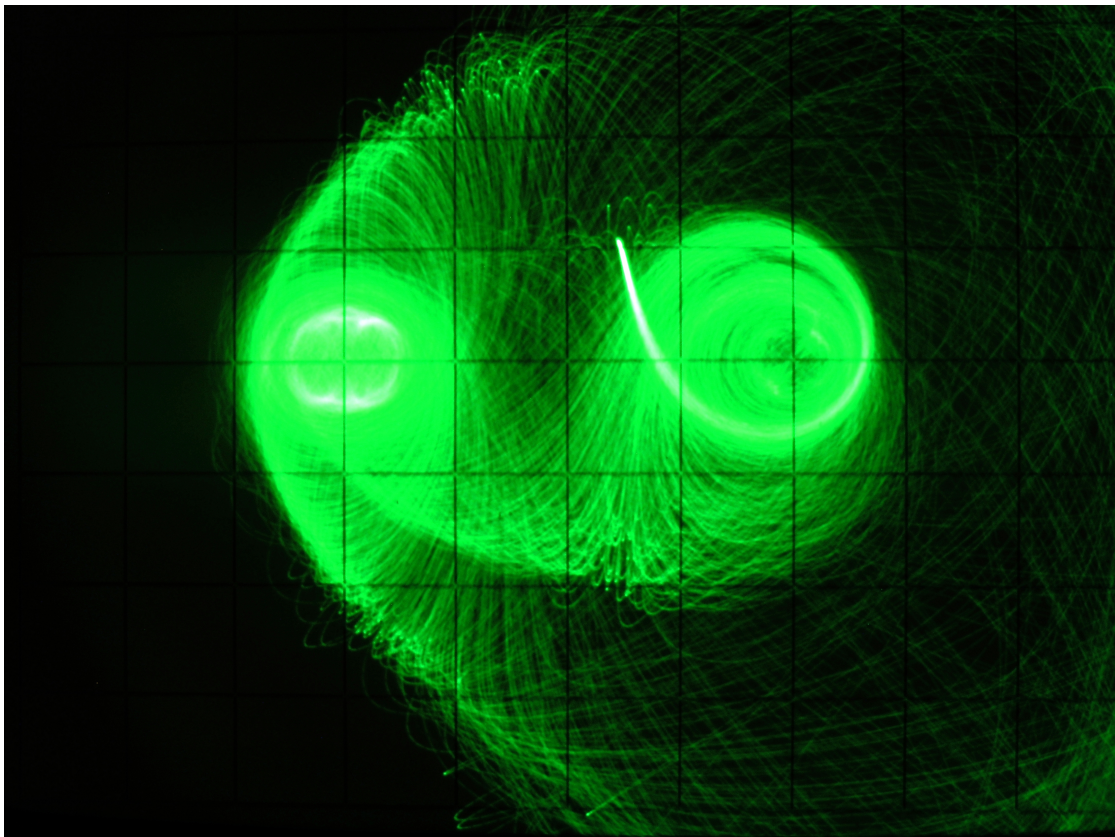


Figure 3: Typical family of satellite trajectories for a small value of G

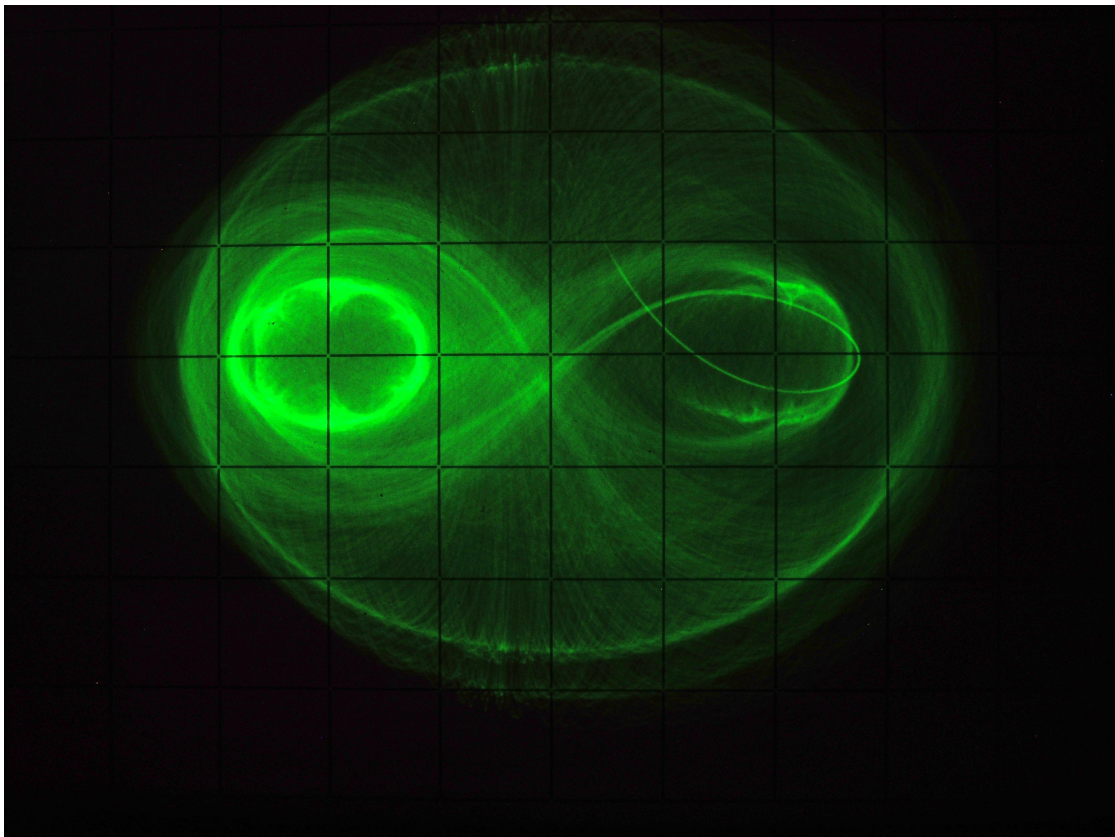


Figure 4: Family of trajectories for a larger value of G