

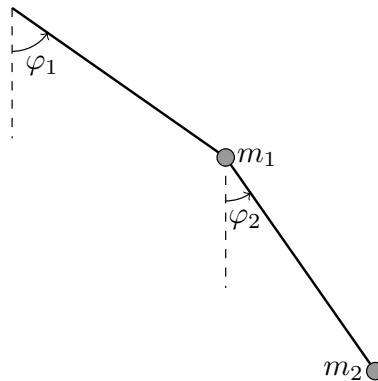


Figure 1: Double pendulum

Simulating a double pendulum

This application note describes the simulation of a double pendulum as shown in figure 1 on an analog computer.¹

1 Equations of motion

The equations of motion will be derived by determining the Lagrangian $L = T - V$ with the two generalized coordinates φ_1 and φ_2 where T represents the total kinetic energy while V is the potential energy of the system. This potential energy is just

$$V = -g((m_1 + m_2)l_1 \cos(\varphi_1) + m_2 l_2 \cos(\varphi_2)) \quad (1)$$

¹This has also been done by [MAHREHOLTZ, 1968, pp. 159–165] which served as an inspiration for this application note, especially the display circuit.

with m_1 and m_2 representing the masses of the two bobs mounted on the tips of the two pendulum rods (which themselves are assumed as being weightless). As always, g represents the gravitational acceleration. To derive the total kinetic energy the positions of the pendulum arm tips (x_1, y_1) and (x_2, y_2) are required:

$$\begin{aligned}x_1 &= l_1 \sin(\varphi_1) \\y_1 &= l_1 \cos(\varphi_1) \\x_2 &= l_1 \sin(\varphi_1) + l_2 \sin(\varphi_2) \\y_2 &= l_1 \cos(\varphi_1) + l_2 \cos(\varphi_2)\end{aligned}$$

The first derivatives of these with respect to time are

$$\begin{aligned}\dot{x}_1 &= \dot{\varphi}_1 l_1 \cos(\varphi_1), \\ \dot{y}_1 &= -\dot{\varphi}_1 l_1 \sin(\varphi_1), \\ \dot{x}_2 &= \dot{\varphi}_1 l_1 \cos(\varphi_1) + \dot{\varphi}_2 l_2 \cos(\varphi_2) \text{ and} \\ \dot{y}_2 &= -\dot{\varphi}_1 l_1 \sin(\varphi_1) - \dot{\varphi}_2 l_2 \sin \varphi_2.\end{aligned}$$

The kinetic energy of the system is then

$$T = \frac{1}{2} (m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_s(\dot{x}_2^2 + \dot{y}_2^2)) \quad (2)$$

with

$$\begin{aligned}\dot{x}_1^2 &= \dot{\varphi}_1^2 l_1^2 \cos^2(\varphi_1) \\ \dot{y}_1^2 &= \dot{\varphi}_1^2 l_1^2 \sin^2(\varphi_1) \\ \dot{x}_2^2 &= \dot{\varphi}_1^2 l_1^2 \cos^2(\varphi_1) + 2\dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \cos(\varphi_1) \cos(\varphi_2) + \dot{\varphi}_2^2 l_2^2 \cos^2(\varphi_2) \\ \dot{y}_2^2 &= \dot{\varphi}_1^2 l_1^2 \sin^2(\varphi_1) + 2\dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \sin(\varphi_1) \sin(\varphi_2) + \dot{\varphi}_2^2 l_2^2 \sin^2(\varphi_2)\end{aligned}$$

and thus

$$\begin{aligned} \dot{x}_1^2 + \dot{y}_1^2 &= \dot{\varphi}_1^2 l_1^2 \cos^2(\varphi_1) + \dot{\varphi}_1^2 l_1^2 \sin^2(\varphi_1) \\ &= \dot{\varphi}_1^2 l_1^2 (\cos^2(\varphi_1) + \sin^2(\varphi_1)) \\ &= \dot{\varphi}_1^2 l_1^2 \end{aligned} \quad (3)$$

$$\begin{aligned} \dot{x}_2^2 + \dot{y}_2^2 &= \dot{\varphi}_1^2 l_1^2 (\cos^2(\varphi_1) + \sin^2(\varphi_1)) + \dot{\varphi}_2^2 l_2^2 (\cos^2(\varphi_2) + \sin^2(\varphi_2)) + \\ &\quad 2\dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 (\cos(\varphi_1) \cos(\varphi_2) + \sin(\varphi_1) \sin(\varphi_2)) \\ &= \dot{\varphi}_1^2 l_1^2 + \dot{\varphi}_2^2 l_2^2 + 2\dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \cos(\varphi_1 - \varphi_2) \end{aligned} \quad (4)$$

The Lagrangian L results from equations (1) and (2) with (3) and (4) as

$$\begin{aligned} L &= \frac{1}{2} l_1^2 \dot{\varphi}_1^2 (m_1 + m_2) + \frac{1}{2} m_2 \dot{\varphi}_2^2 l_2^2 + m_2 \dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \cos(\varphi_1 - \varphi_2) + \\ &\quad g(m_1 + m_2) l_1 \cos(\varphi_1) + g m_2 l_2 \cos(\varphi_2). \end{aligned} \quad (5)$$

Now the EULER-LAGRANGE-equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_1} - \frac{\partial L}{\partial \varphi_1} = 0 \quad \text{and} \quad (6)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_2} - \frac{\partial L}{\partial \varphi_2} = 0 \quad (7)$$

have to be solved. The required (partial) derivatives are

$$\begin{aligned} \frac{\partial L}{\partial \varphi_1} &= -m_2 \dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \sin(\varphi_1 - \varphi_2) - g(m_1 + m_2) l_1 \sin(\varphi_1) \\ \frac{\partial L}{\partial \dot{\varphi}_1} &= l_1^2 \dot{\varphi}_1 (m_1 + m_2) + m_2 \dot{\varphi}_2 l_1 l_2 \cos(\varphi_1 - \varphi_2) \end{aligned}$$

$$\begin{aligned}\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_1} &= l_1^2 \ddot{\varphi}_1 (m_1 + m_2) + m_2 l_1 l_2 [\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \\ &\quad - \dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) (\dot{\varphi}_1 - \dot{\varphi}_2)] \\ \frac{\partial L}{\partial \varphi_2} &= m_2 \dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \sin(\varphi_1 - \varphi_2) - g m_2 l_2 \sin(\varphi_2) \\ \frac{\partial L}{\partial \dot{\varphi}_2} &= m_2 \dot{\varphi}_2 l_2^2 + m_2 \dot{\varphi}_1 l_1 l_2 \cos(\varphi_1 - \varphi_2) \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_2} &= m_2 \ddot{\varphi}_2 l_2^2 + m_2 l_1 l_2 [\ddot{\varphi}_1 \cos(\varphi_1 - \varphi_2) \\ &\quad - \dot{\varphi}_1 \sin(\varphi_1 - \varphi_2) (\dot{\varphi}_1 - \dot{\varphi}_2)] .\end{aligned}$$

based on (5). Substituting these into (6) yields

$$\begin{aligned}0 &= l_1^2 \ddot{\varphi}_1 (m_1 + m_2) - m_2 l_1 l_2 \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \\ &\quad - m_2 l_1 l_2 \dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) (\dot{\varphi}_1 - \dot{\varphi}_2) \\ &\quad + m_2 \dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \sin(\varphi_1 - \varphi_2) + g l_1 (m_1 + m_2) \sin(\varphi_1) .\end{aligned}$$

Expanding $(\dot{\varphi}_1 - \dot{\varphi}_2)$ yields

$$\begin{aligned}0 &= l_1^2 \ddot{\varphi}_1 (m_1 + m_2) + m_2 l_1 l_2 \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \\ &\quad m_2 l_1 l_2 \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) + g l_1 (m_1 + m_2) \sin(\varphi_1) .\end{aligned}$$

Dividing by $l_1^2 (m_1 + m_2)$ results in

$$\begin{aligned}0 &= \ddot{\varphi}_1 + \frac{m_2}{m_1 + m_2} \frac{l_2}{l_1} \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \\ &\quad + \frac{m_2}{m_1 + m_2} \frac{l_2}{l_1} \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) + \frac{g}{l_1} \sin(\varphi_1)\end{aligned}$$

which can be further simplified as

$$0 = \ddot{\varphi}_1 + \frac{m_2}{m_1 + m_2} \frac{l_2}{l_1} [\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2)] + \frac{g}{l_1} \sin(\varphi_1). \quad (8)$$

Proceeding analogously, (7) yields

$$0 = \ddot{\varphi}_2 + \frac{l_1}{l_2} [\ddot{\varphi}_1 \cos(\varphi_1 - \varphi_2) - \dot{\varphi}_1^2 \sin(\varphi_1 - \varphi_2)] + \frac{g}{l_2} \sin(\varphi_2). \quad (9)$$

To simplify things further it will be assumed that $m_1 = m_2 = 1$ and $l_1 = l_2 = 1$ in arbitrary units yielding the following two equations of motion of the double pendulum based on (8) and (9):

$$\ddot{\varphi}_1 = -\frac{1}{2} [\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) + 2g \sin(\varphi_1)] \quad (10)$$

$$\ddot{\varphi}_2 = -[\ddot{\varphi}_1 \cos(\varphi_1 - \varphi_2) - \dot{\varphi}_1^2 \sin(\varphi_1 - \varphi_2) + g \sin(\varphi_2)] \quad (11)$$

2 Implementation

The implementation of equations (10) and (11) is straightforward as shown in figures 2 and 3. Both circuits require the functions $\sin(\varphi_1 - \varphi_2)$, $\cos(\varphi_1 - \varphi_2)$, $\sin(\varphi_1)$, and $\sin(\varphi_2)$, which are generated by three distinct sub-circuits such as the one shown in figure 4. Each of these circuits yields sine and cosine of an angle based on the first time-derivative of this angle, so the simulation is not limited to a finite range for the two angles φ_1 and φ_2 allowing the pendula to roll over.

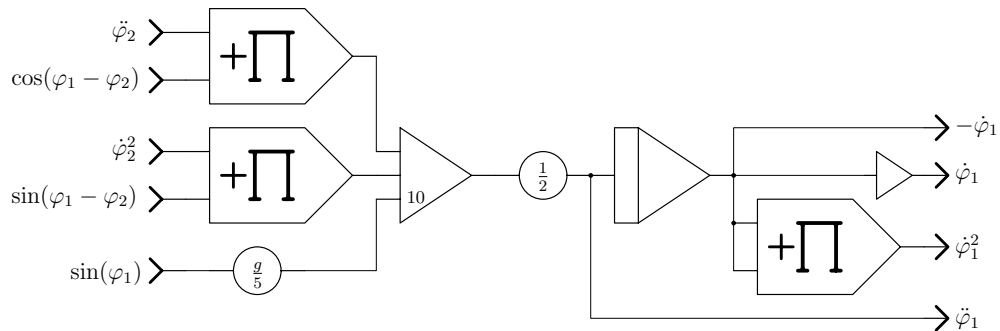


Figure 2: Implementation of equation (10)

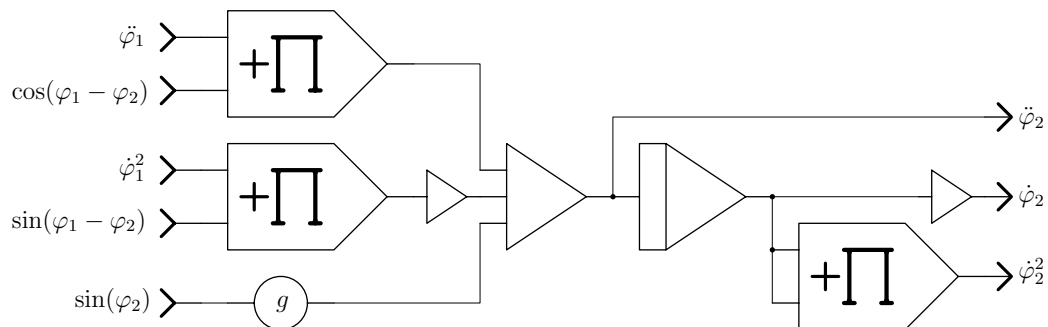


Figure 3: Implementation of equation (11)

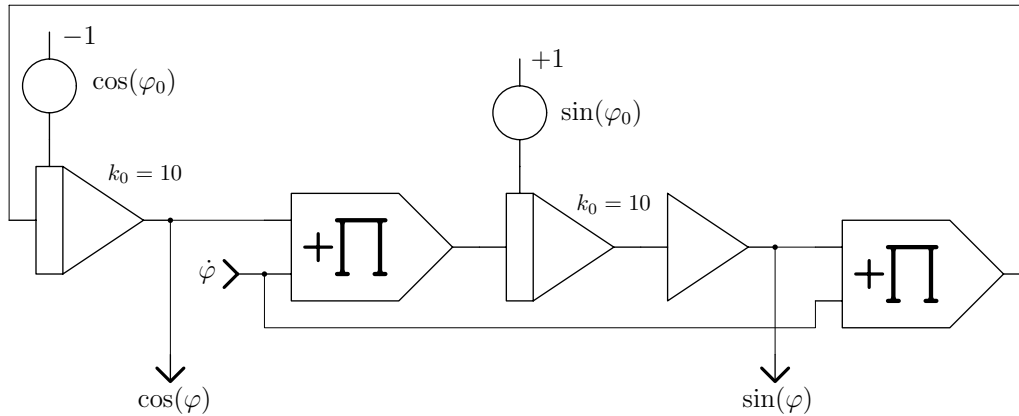


Figure 4: Generating $\sin(\varphi)$ and $\cos(\varphi)$

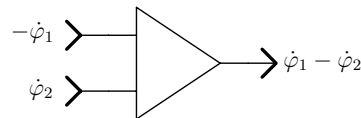


Figure 5: Computing $\dot{\varphi}_1 - \dot{\varphi}_2$

It should be noted that each of the two integrators of each of these three harmonic function generators has a potentiometer connected to its respective initial value input. When a simulation run with given initial values for $\varphi_1(0)$ and $\varphi_2(0)$ is to be started, these potentiometers have to be set to the values $\cos(\varphi_1(0))$, $\sin(\varphi_1(0))$ and $\cos(\varphi_2(0))$, $\sin(\varphi_2(0))$, respectively.

Since $\dot{\varphi}_1$ and $\dot{\varphi}_2$ are readily available from the circuits shown in figures 2 and 3, two of these harmonic function generators can be directly fed with these values. The input for the third function generator is generated by a two-input summer as shown in figure 5.

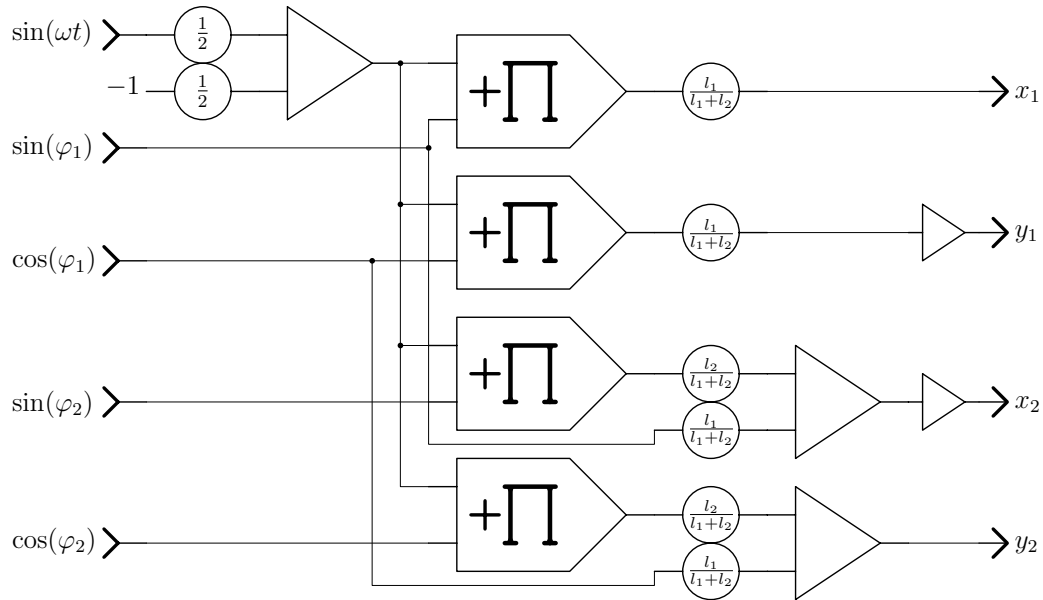


Figure 6: Display circuit for the double pendulum

With $\dot{\varphi}_1$ and $\dot{\varphi}_2$ and thus $\sin(\varphi_1)$, $\cos(\varphi_1)$, and $\sin(\varphi_2)$, $\cos(\varphi_2)$ readily available, the double pendulum can be displayed on an oscilloscope featuring two separate x/y -inputs by means of the circuit shown in figure 6. To yield a flicker-free display, this circuit requires a high-frequency input $\sin(\omega t)$ which can be obtained by two integrators in series with a summer, the output of which is fed back into the first integrators. This circuit is not shown here but can be found in [ULMANN 2017, p. 67].

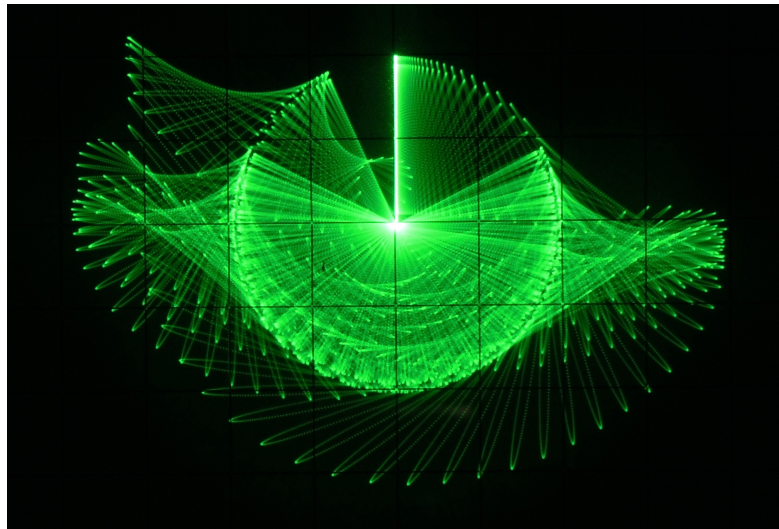


Figure 7: Long-term exposure of a double pendulum simulation run

3 Results

Figure 7 shows a long-term exposure of the movements of the double pendulum starting as an inverted pendulum with its first pendulum rod pointing upwards while the second one points downwards.

This simulation requires a total of ten integrators, 15 summers, 16 multipliers, and 17 coefficient potentiometers.

References

[MAHRENHOLTZ, 1968] OSKAR MAHRENHOLTZ, *Analogrechnen in Maschinenbau und Mechanik*, BI Hochschultaschenb"ucher, 1968



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