

The Duffing Oscillator

A *Duffing equation* describes any oscillator featuring a cubic stiffness term, i. e., a nonlinear elasticity, such as

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = 0.$$

It is named after *Georg Wilhelm Christian Caspar Duffing* (1861 – 1944), a German engineer and inventor who wrote a seminal work on forced oscillations in 1918 (cf. [DUFFING 1918]).

This oscillator exhibits a nice chaotic behaviour when driven by an external forcing function i. e. in the form of

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega_0 t).$$

Thorough theoretical treatments of this particular oscillator may be found in [KORSCH et al. 2008], [HÖHLER 1988], and [CHANG 2017]. Apart from its interesting mathematical properties it is a nice dynamic system which can be implemented on an analog computer without complicated scaling. It invites to playing with parameters to achieve all kinds of nice phase space plots.

Figure 1 shows the implementation of the forcing function $f(t) = \gamma \cos(\omega_0 t)$. To achieve a minimum of harmonic distortion, the two amplitude limiting Zener diodes are connected to an integrator input with weight 1. ε introduces a tiny positive feedback signal to avoid a decreasing amplitude, while γ controls the amplitude of the output signal.

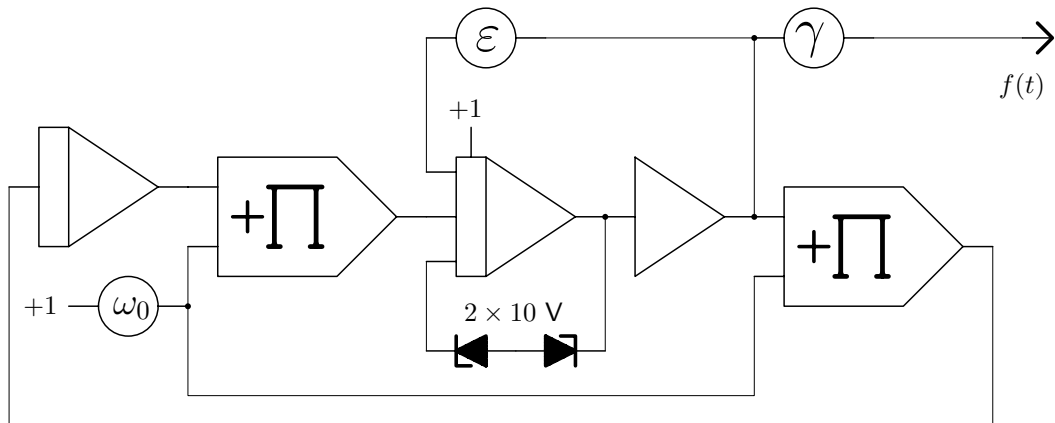


Figure 1: Forcing function $f(t)$

Figure 2 shows the straightforward implementation of the DUFFING oscillator. β was scaled down by a factor of $\frac{1}{10}$ to allow for values of β up to 10.

A good parameter set is

$$\begin{aligned}\alpha &= 1, \\ \beta &= 5, \\ \gamma &= 1, \\ \delta &= 0.02, \\ \omega_0 &= 1, \text{ and} \\ \varepsilon &= \text{as small as possible.}\end{aligned}$$

Varying the parameters manually shows the very rich behaviour of this forced oscillator which even shows chaotic characteristics. The effect of ω_0 is quite distinct. Two typical phase space plots generated by plotting x against $-\dot{x}$ are shown in figure 3. To get a wider range for γ the

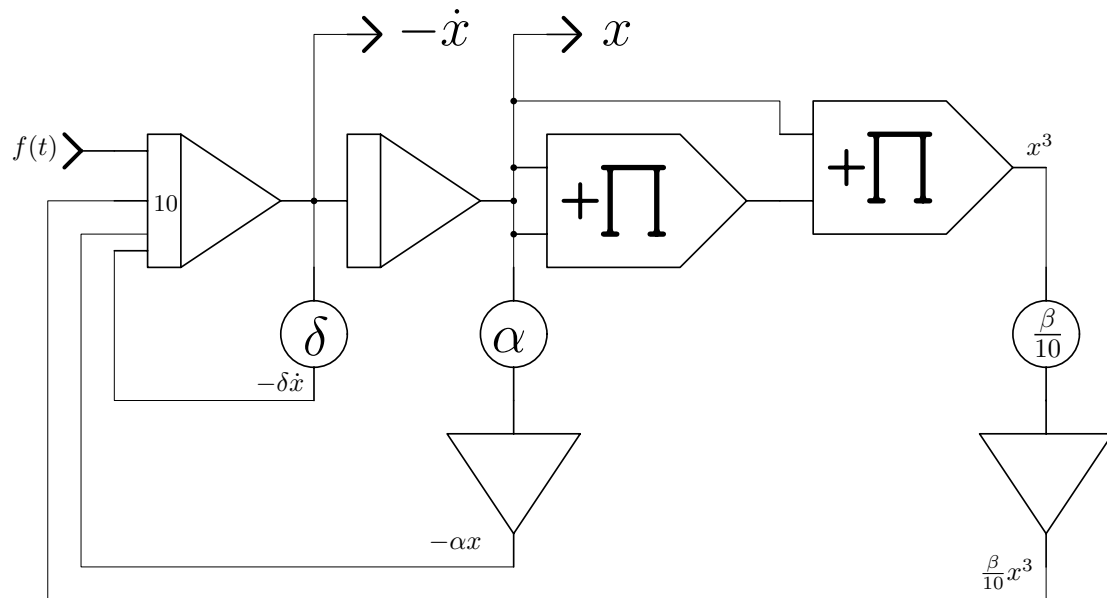


Figure 2: DUFFING oscillator

output $f(t)$ may also be connected to an input with weight 10 on the DUFFING oscillator sub-circuit.

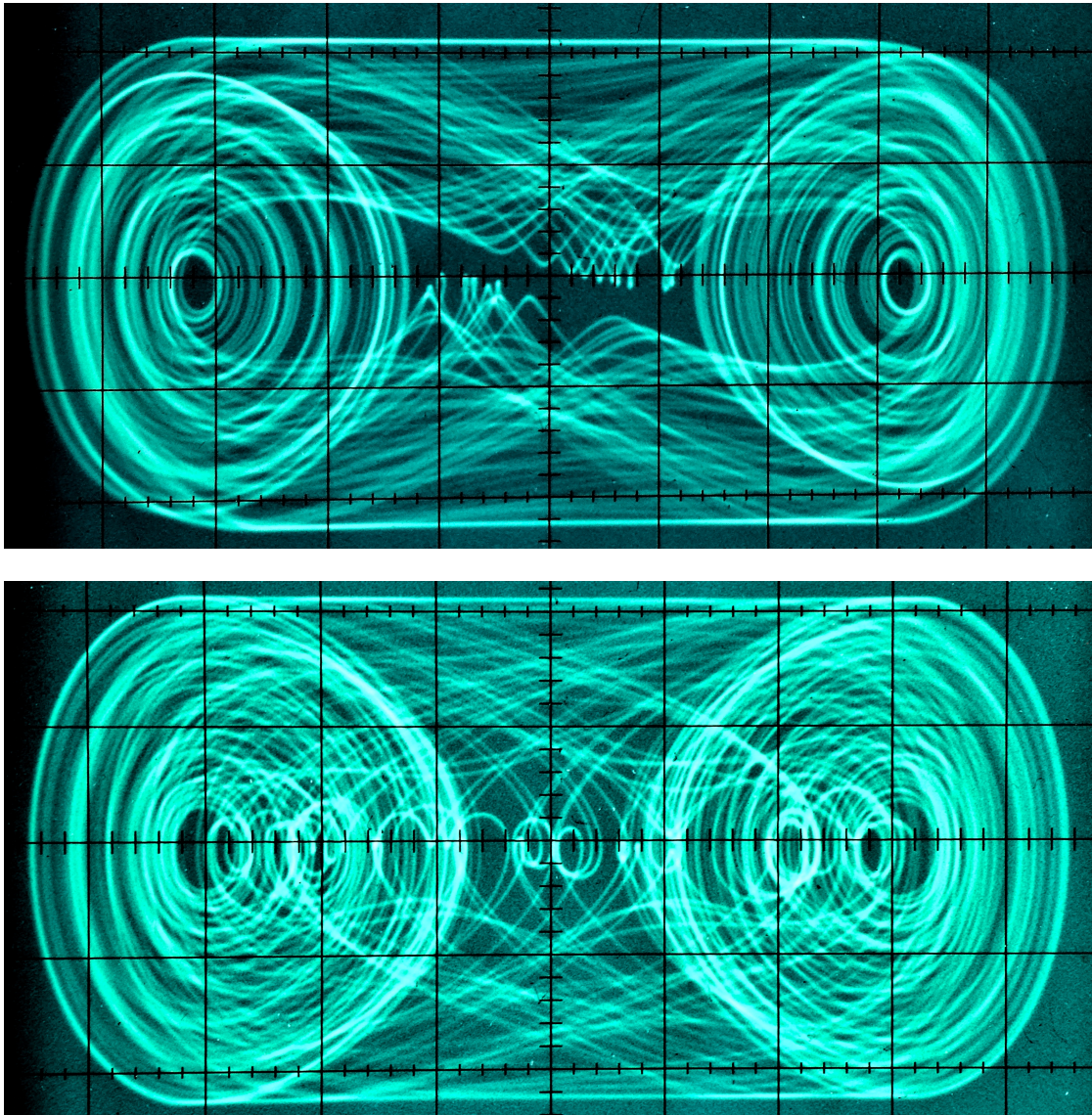


Figure 3: Two phase space plots showing the chaotic behaviour of the DUFFING oscillator

References

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- [KORSCH et al. 2008] HANS JÜRGEN KORSCH, HANS-JÖRG JODL, TIMO HARTMANN, “The Duffing Oscillator”, in *Chaos – A Program Collection for the PC*, Springer, 2008, pp. 157–184