

The EULER spiral

The first example in the beautiful book [HAVIL(2019), pp. 1 ff.] is the *EULER spiral*, which is a *clothoid*, i.e., a curve with a curvature depending linearly on its arc length. Clothoids play a significant role in road and railway track construction. Imagine driving along a straight road, approaching a curve. If the straight part of the road was directly connected to an arc of a circle forming the curve, one would have to go from a steering angle of zero to a non-zero steering angle instantaneously, which is impracticable even at rather low speeds. Therefore, a safe curve in a road has a curvature that starts at zero, rises to a maximum and then falls back to zero again when the next straight segment of the road is entered. This can be achieved by using clothoids and many curves encountered on highways and railway tracks are based on this type of curve.

Generally, a curve parameterized by some functions $x(t)$, $y(t)$ has a slope

$$m = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}, \quad (1)$$

an arc length

$$l = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2} dt, \quad (2)$$

and a curvature

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\sqrt{(\dot{x}^2 + \dot{y}^2)^3}}. \quad (3)$$

Using the general parameterization

$$x(t) = \int_0^T \cos(f(t)) \, dt \text{ and}$$

$$y(t) = \int_0^T \sin(f(t)) \, dt$$

yields the following time derivatives:

$$\begin{aligned} \dot{x} &= \cos(f(t)) & \dot{y} &= \sin(f(t)) \\ \ddot{x} &= -\dot{f} \sin(f(t)) & \ddot{y} &= \dot{f} \cos(f(t)) \end{aligned}$$

Inserting these into (1), (2), and (3) yields

$$m = \frac{\sin(f(t))}{\cos(f(t))} = \tan(f(t)),$$

$$l = \int_0^T \sqrt{\cos^2(f(t)) + \sin^2(f(t))} \, dt = \int_0^T dt = T, \text{ and}$$

$$\kappa = \frac{\dot{f} (\cos^2(f(t)) + \sin^2(f(t)))}{\sqrt{(\cos^2(f(t)) + \sin^2(f(t)))^3}} = \dot{f}.$$

The EULER spiral satisfies $\kappa = t$, i. e. $\dot{f} = t$ yielding $f(t) = \frac{t^2}{2}$ from

which its parameterization

$$x(t) = \int_0^T \cos\left(\frac{t^2}{2}\right) dt \text{ and}$$
$$y(t) = \int_0^T \sin\left(\frac{t^2}{2}\right) dt$$

follows. This looks pretty straightforward to implement on an analog computer but classic function generators for $\sin(\varphi)$ and $\cos(\varphi)$ have a limited interval for φ , typically $[-\frac{\pi}{2}, \frac{\pi}{2}]$. If more than a very short segment of the EULER spiral is to be generated by an analog computer, a trick must be employed. The idea is to use a quadrature generator¹ which gets $\dot{\varphi}$ instead of φ as its input. Such a generator, basically consisting of two integrators, two multipliers (to introduce $\dot{\varphi}$ into the underlying DEQ) and an inverter is at the heart of the circuit.

Figure 1 shows the analog computer program for generating an EULER spiral. The quadrature output signals $\cos\left(\frac{t^2}{2}\right)$ and $\sin\left(\frac{t^2}{2}\right)$ are fed into integrators yielding $x(t)$ and $y(t)$, while $\dot{\varphi}$ is obtained by another integrator generating a linear ramp from -1 to $+1$. Figures 2 and 3 show the actual implementation of the program on *THE ANALOG THING* and a typical curve display. The analog computer was set to repetitive operation and the operation time was set so that the output of the integrator yielding $\dot{\varphi}$ is a ramp from -1 to $+1$ corresponding to roughly 20 ms. Typical values for coefficients are shown in table 1.

¹Cf. [ULMANN(2020), pp. 60 ff].

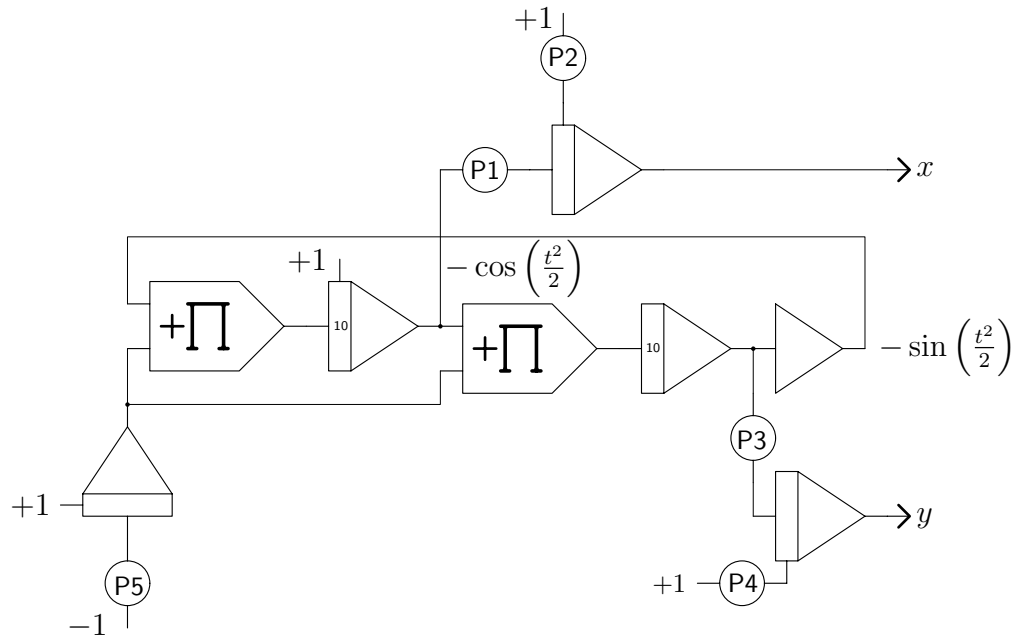


Figure 1: Analog computer program for generating an EULER spiral

Increasing the operating time slowly allows to show the growth of the EULER spiral on the oscilloscope screen. Ideally, all integrators are run with the highest time scale factor k_0 selected.²

²On THE ANALOG THING this corresponds to $k_0 = 10^3$.

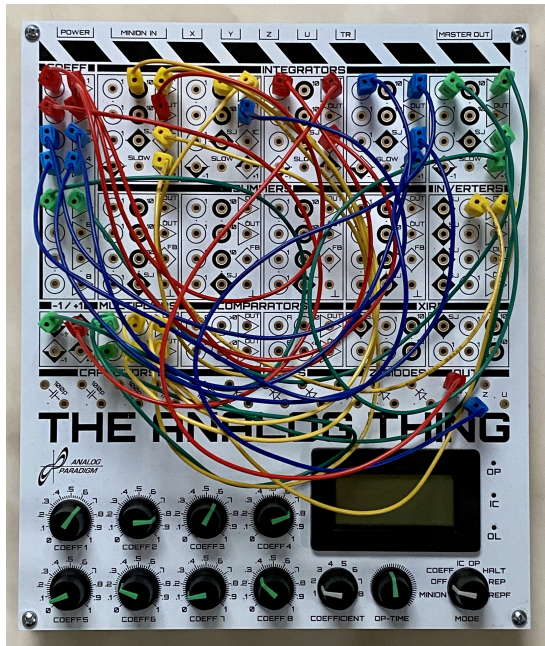


Figure 2: Setup for generating an EULER spiral on THE ANALOG THING

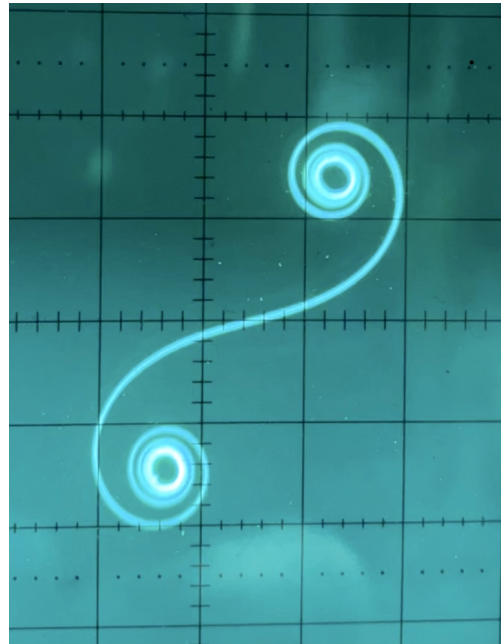
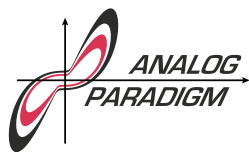


Figure 3: Typical output of the EULER spiral program

Coefficient	Description	Typical value
P1	x scaling	0.6
P2	x shift	0.87
P3	y scaling	0.6
P4	y shift	0.75
P5	ramp slope	0.1

Table 1: Coefficient values



Analog Computer Applications

References

- [HAVIL(2019)] JULIAN HAVIL, *Curves for the Mathematically Curious – an Anthology of the Unpredictable, Historical, Beautiful, and Romantic*, Princeton University Press, 2019
- [ULMANN(2020)] BERND ULMANN, *Analog and Hybrid Computer Programming*, DeGruyter, 2020