

A chaotic system based on $|x|$

An interesting class of systems exhibiting chaotic behavior are *jerk systems*, autonomous differential equations of third order of the general form

$$\ddot{x} = J(x, \dot{x}, \ddot{x}).$$

Here, $\ddot{x}$ is the *jerk*, i. e., the change in acceleration when the equation is assumed to describe a mechanical system, while $J(x, \dot{x}, \ddot{x})$ is the *jerk function*, respectively. The simplest such system has been described in [SPROTT 1997] and is of the form

$$\ddot{x} + A\ddot{x} - \dot{x}^2 + x = 0.$$

Based on this, [LINZ et al. 1999] introduced a jerk system employing an absolute value function:¹

$$\ddot{x} + A\ddot{x} + \dot{x} - |x| + 1 = 0. \quad (1)$$

With $A = 0.6$, all derivatives of x are well within $[-3, 3]$ and can be scaled with a common scaling factor $\lambda = \frac{1}{3}$ which cancels out in all terms but the constant 1 which gets scaled down to $\frac{1}{3}$. The corresponding analog computer program is shown in figure 1.² The necessary absolute value function can be implemented either with a comparator or with the circuit shown in figure 2.³ The open amplifier on the left is used to implement an idealized diode while the amplifier on the right sums

¹Letting $\dot{x} = y$ and $\dot{y} = z$ this system can be written as three coupled differential equations with $\dot{z} = -Az - y + |x| - 1$ which might look more familiar to the analog computer programmer.

²[KIERS et al. 2003] describes an electronic implementation of this system.

³See [ULMANN 2023, pp. 82 f.].

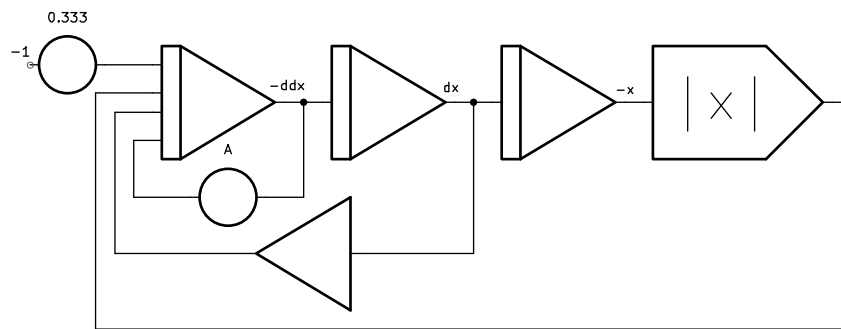


Figure 1: Implementation of equation (1)

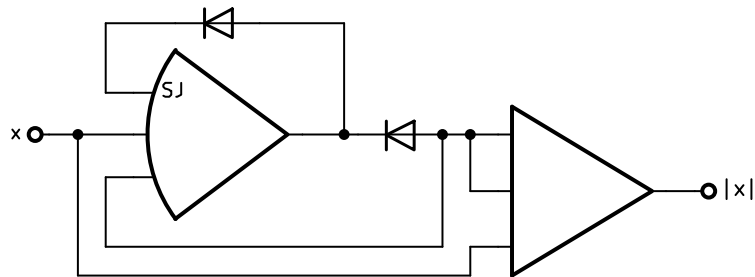


Figure 2: Implementation of $|x|$

twice the output of this ideal diode to one times the input signal. The resulting phase space plots are shown in figure 3.

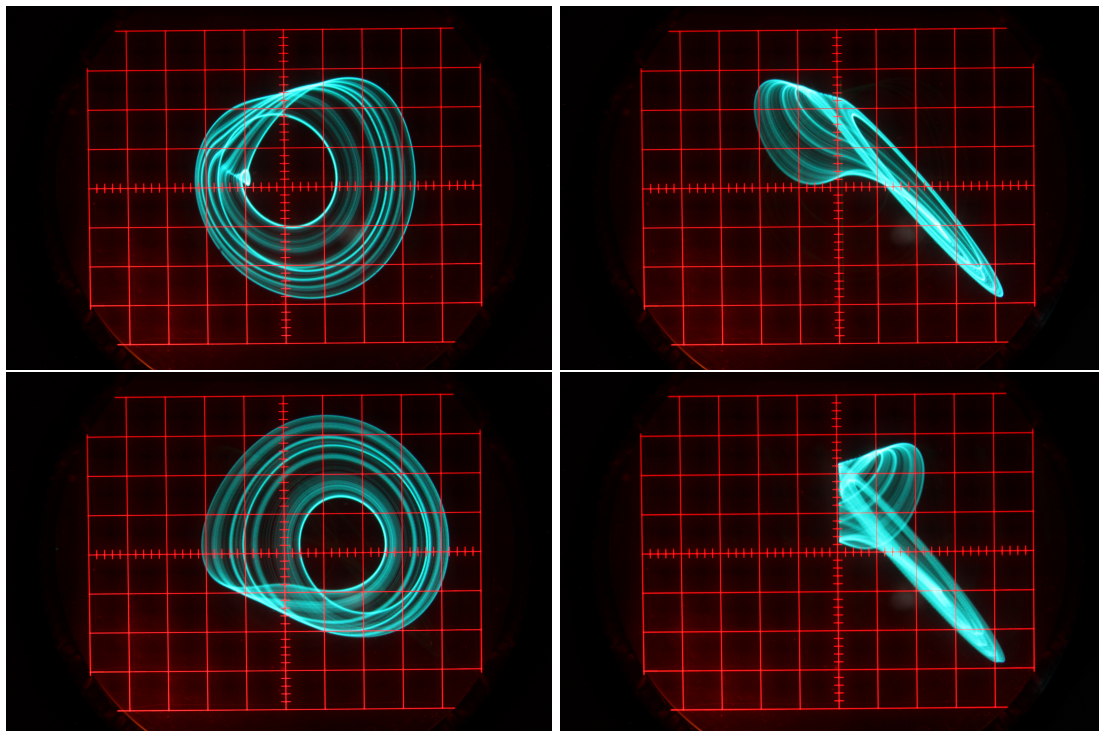


Figure 3: Phase space plots of the system (1) – from left to right and top to bottom the phase space plots show $-\ddot{x}/\dot{x}$, $-\ddot{x}/-x$, $\dot{x}/-x$, and $-\ddot{x}/|x|$



Analog Computer Applications

References

- [KIERS et al. 2003] KEN KIERS, TIM KLEIN, JEFF KOLB, STEVE PRICE, J. C. SPROTT, "Chaos in a nonlinear analog computer", in *International Journal of Bifurcation and Chaos*, Vol. 14, No. 8, 2004, pp. 2867–2873
- [LINZ et al. 1999] STEFAN J. LINZ, J. C. SPROTT, "Elementary chaotic flow", in *Physics Letters A*, 256, 1999, pp. 240–245
- [SPROTT 1997] J. C. SPROTT, "Simplest dissipative chaotic flow", in *Physics Letters A*, 228, 1997, pp. 271–273
- [ULMANN 2023] BERND ULMANN, *Analog and Hybrid Computer Programming*, 2nd edition, DeGruyter, 2023