

Damped pendulum with external forcing function

1 Introduction

A damped pendulum subjected to an external (harmonic) excitation function $e(t)$ exhibits chaotic behavior and can be easily simulated on an analog computer as the following application note shows.¹

We assume a punctiform mass m on a massless pendulum rod of length r . The angle between this rod and a vertical line crossing its hinge is denoted by φ . g is the acceleration of gravity, β represents the damping factor, while A is the amplitude of the excitation function.

With I denoting the rotational inertia, we have

$$I = mr^2.$$

Since the sum of all forces in a closed system must be zero, it follows that

$$I\ddot{\varphi} = \sum_i \tau_i, \quad (1)$$

where the τ_i denote the various torques acting in the system:

$\tau_g = -rmg \sin(\varphi)$	torque due to g
$\tau_\beta = -\beta\dot{\varphi}$	torque due to β
$\tau_e = A \cos(\omega t)$	torque due to excitation

¹This note was motivated by <https://www.myphysicslab.com/pendulum/pendulum/chaotic-pendulum-en.html>.

Applying these to (1) results in

$$mr^2\ddot{\varphi} = -rmg \sin(\varphi) - \beta\dot{\varphi} + A \cos(\omega t)$$

which can be rearranged to

$$\ddot{\varphi} = -\frac{g}{r} \sin(\varphi) + \frac{A \cos(\omega t) - \beta\dot{\varphi}}{mr^2}.$$

As the highest derivative is already isolated on the left-hand side this form is ideally suited to derive an analog computer setup using KELVIN's feedback technique.

2 Excitation function

First of all we need a harmonic excitation function with a variable frequency. Basically such a problem can always be reduced to solving a second order ordinary differential equation of the form $\ddot{y} = -y$. Due to unavoidable imperfections of real integrators we will add some amplitude stabilization as shown in figure 1.

3 Sine function

If we allow large displacement angles, the sine-term in τ_g can no longer be approximated by $\sin(\varphi) \approx \varphi$ as is normally done in school. Instead a "real" sine generator is required. This could be implemented by a diode function generator, approximating the sine by a polygon with rather good accuracy but this approach will fail as soon as the angle leaves the interval $[-\pi : \pi]$ (or any other fixed interval on which the function

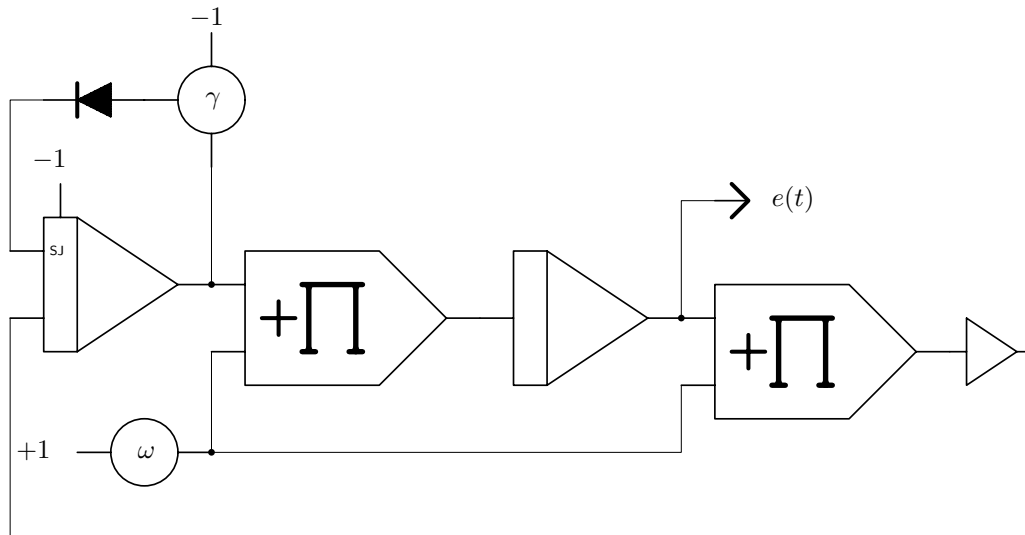


Figure 1: Forcing function $e(t)$

generator will yield useful results) which will invariable happen when our pendulum rolls over due to the external forcing function. Thus we resort to a classic trick: Generating the sine not based on φ but rather on $\dot{\varphi}$ instead.

This makes it possible to use a second circuit of the form shown in figure 1. Here, the two multipliers are connected to $\dot{\varphi}$ which is readily available from the pendulum subcircuit as shown in figure 2. The output (corresponding to $e(t)$ in figure 1) then yields the value $\sin(\int \dot{\varphi} dt)$. So all in all we need two instances of the circuit shown in figure 1.

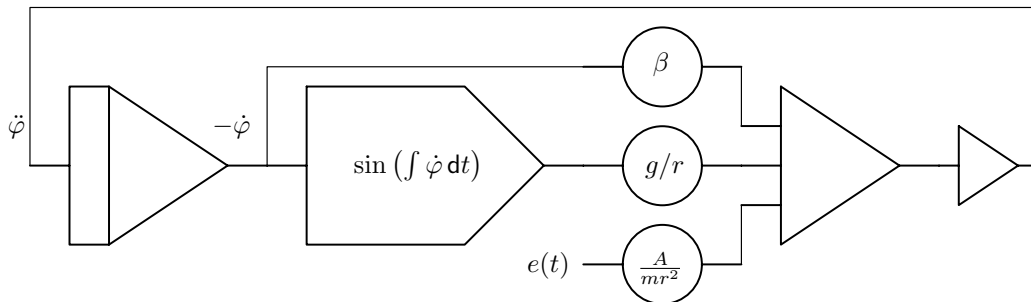


Figure 2: Computer setup for pendulum simulation

4 Pendulum simulation

Using these two subcircuits we can now setup the pendulum simulation itself as shown in figure 2.

5 Results

This setup invites to play with the parameters, β , A/mr^2 , g/r , and ω . The oscilloscope screen image shown in figure 3 is a typical phase-space plot of $\ddot{\varphi}$ against $-\dot{\varphi}$.

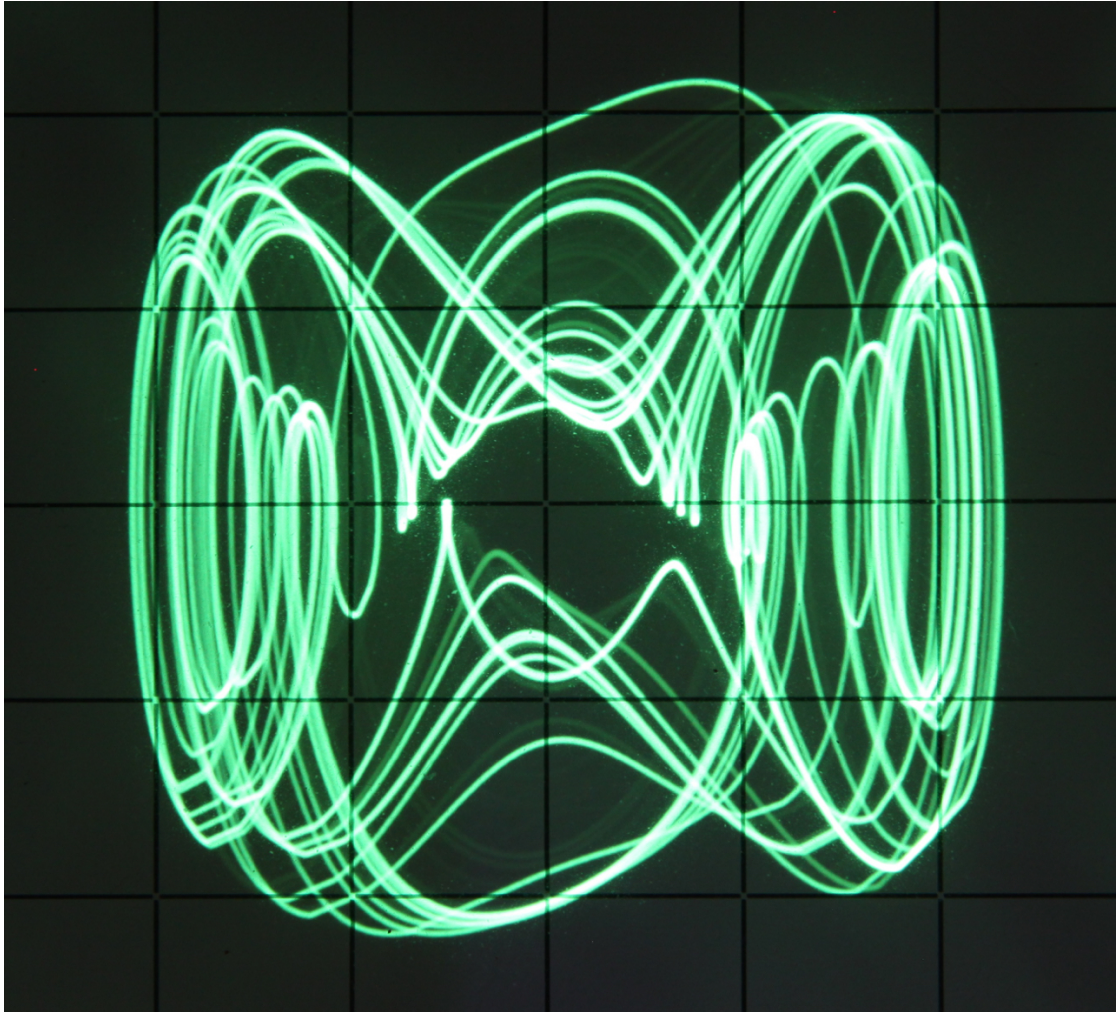


Figure 3: Typical simulation result showing chaotic behavior