

HAUSNER's parametric technique

A few weeks ago I stumbled over an intriguing technique described by ARTHUR HAUSNER¹ (see [HAUSNER 1962]). In analog/hybrid computer programming differential equations such as LEGENDRE's²

$$(1 - x^2)y''(x) - 2xy'(x) - n(n + 1)y(x) = 0$$

are a challenge since the naïve approach results in a fraction on the right hand side of

$$y''(x) = \frac{2xy'(x) - n(n + 1)y(x)}{1 - x^2}$$

which may be problematic for $x \rightarrow 1$.³

HAUSNER's technique is based on the chain rule: Given some DEQ of the form

$$\frac{d^n y}{dx^n} = y^{(n)} = \frac{p(x, y, y', \dots, y^{(n-1)})}{q(x, y, y', \dots, y^{(n-1)})}$$

this can be transformed into a set of DEQs of first order by defining y_i as $y_i = y^{(i)}$, i. e., $\frac{dy}{dx} = y_1$ and $\frac{dy_i}{dx} = y_{i+1}$ for $1 < i < n - 1$ and thus

$$\frac{dy_{n-1}}{dx} = y_n = \frac{p(x, y, y', \dots, y^{(n-1)})}{q(x, y, y', \dots, y^{(n-1)})}.$$

Introducing a new parameter τ and differentiating with respect to it

¹27.11.1929–15.10.2020, see <https://www.santacruzsentinel.com/obituaries/arthur-hausner/>, retrieved 03.08.2026

²ADRIEN-MARIE LEGENDRE, 18.09.1752–09.01.1833

³One way of solving this has been shown in application note #38: https://analogparadigm.com/downloads/alpaca_38.pdf, retrieved 03.08.2026.

yields

$$\frac{dy}{dx} = \frac{\frac{dy}{d\tau}}{\frac{dx}{d\tau}} = \frac{\dot{y}}{\dot{x}} = y_1,$$

i. e., $\dot{y} = y_1 \dot{x}, \dots, \dot{y}_i = y_{i+1} \dot{x}$.

This technique can now be applied to LEGENDRE's DEQ as an example. We have

$$y'' = \dot{y}_1 = \frac{p(x, y, y', \dots, y^{(n-1)})}{q(x, y, y', \dots, y^{(n-1)})} \dot{x} = \frac{2xy_1 - \eta y}{1 - x^2} \dot{x}$$

with $\eta := n(n+1), n \in \mathbb{N}$. Now the “nasty” denominator $1 - x^2$ can be eliminated by setting $\dot{x} := 1 - x^2$. This results in the following set of three coupled DEQs:

$$\dot{x} = 1 - x^2$$

$$\dot{y} = y_1 \dot{x}$$

$$\dot{y}_1 = 2xy_1 - \eta y$$

Voilà – the fraction is completely gone! Of course, there is no such thing like a free lunch – now three multipliers are required to implement this system. x and y do not even need scaling as long as x is restricted to, e. g., $[0, 1]$. However, y_1 needs scaling. Assuming $n \leq 5$, $\max(\eta) = 30$, suggesting a scaling factor of $\lambda_{y_1} = 10^{-2}$ yields

$$\dot{x} = 1 - x^2, \tag{1}$$

$$\dot{y} = 10\hat{y}_1 \dot{x}, \text{ and} \tag{2}$$

$$\dot{\hat{y}}_1 = 2xy_1 - \hat{\eta}y \tag{3}$$

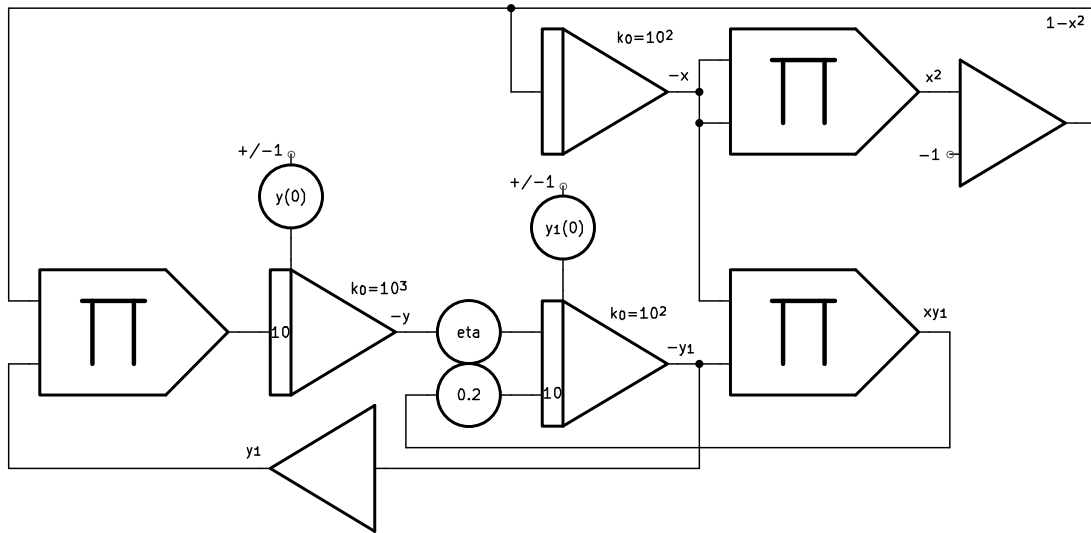


Figure 1: Analog computer program for equations (1), (2), and (3)

with $\hat{\eta} = \frac{\eta}{100}$, $k_{0_x} = k_{0_{y_1}} = 10^2$ and $k_{0_y} = 10^3$. This last k_{0_y} together with the additional weight 10 in the second equation compensates for λ_{y_1} . The resulting analog computer setup is shown in figure 1.

To generate the various Legendre polynomials as solution for this set of coupled DEQs, three parameters have to be taken into account as shown in table 1: η , $y(0)$ and $y_1(0)$. Due to the above scaling factor of $\lambda_{y_1} = 10^2$ the actual parameters are $\hat{\eta} = \frac{\eta}{100}$ and $\hat{y}_1(0) = \frac{y_1(0)}{100}$.

Due to the transformation $\dot{x} = 1 - x^2$ the resulting function can be displayed as a phase space plot by plotting x against y .⁴ Figure 2 shows the result for $n = 5$, i. e., $\hat{\eta} = 0.3$, $y(0) = 0$, and $\hat{y}_1(0) = -0.0188$.

⁴This might require additional inverters.

n	$P_n(t)$	$y(0)$	$\dot{y}(0)$
1	t	0	1
2	$\frac{3}{2}x^2 - \frac{1}{2}$	$-\frac{1}{2}$	0
3	$\frac{5}{2}x^3 - \frac{3}{2}x$	0	$-\frac{3}{2}$
4	$\frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$	$\frac{3}{8}$	0
5	$\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$	0	$\frac{15}{8}$

Table 1: The first five $P_n(t)$ solving LEGENDRE's DEQ with the resulting initial conditions

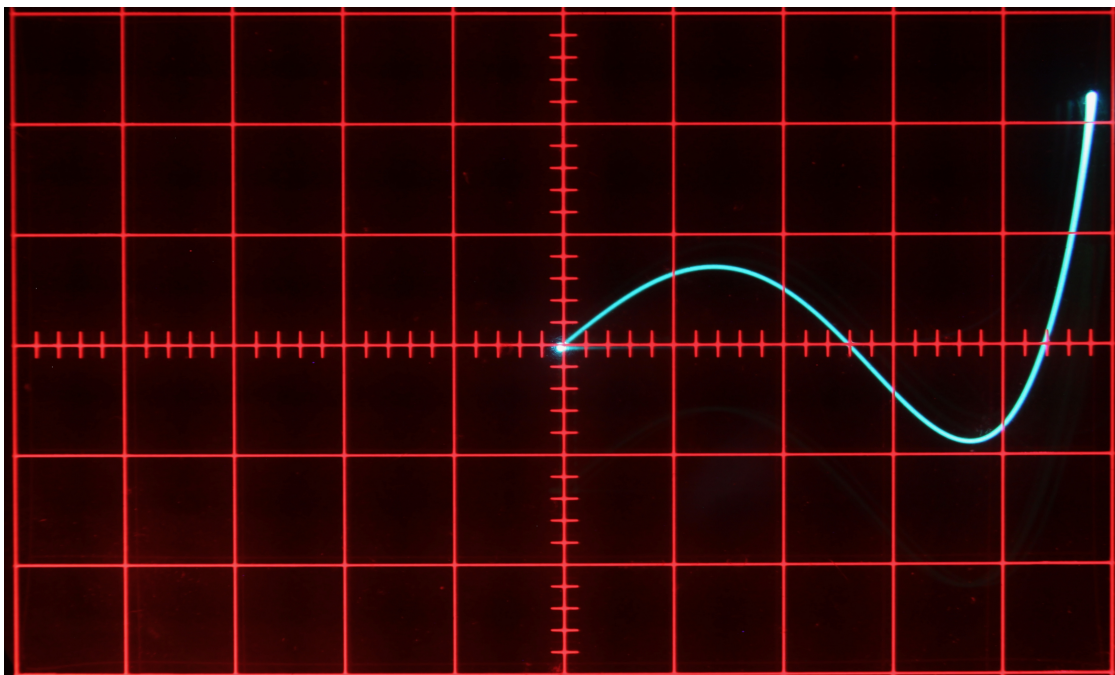


Figure 2: Solution for the LEGENDRE DEQ for $n = 5$



Analog Computer Applications

References

- [HAUSNER 1962] ARTHUR HAUSNER, "Parametric Techniques for Eliminating Division and Treating Singularities in Computer Solutions of Ordinary Differential Equations", in *IRE Transactions on Electronic Computers*, February 1962, pp. 42–45